Solutions

Exam 2
Sections 2.1-2.5 and 3.1-3.4

Name:________________________

Do not write your name on any other page. Answer the following questions. *Answers without proper evidence of knowledge will not be given credit.* Make sure to make reasonable simplifications.

Show your work!

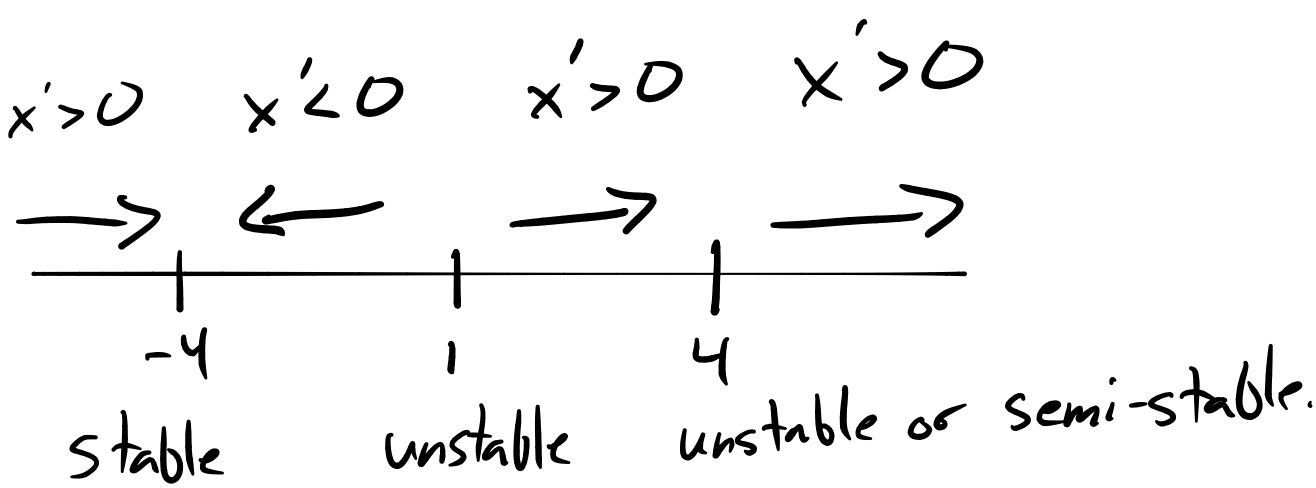
1. (10 points) Draw the phase diagram for the autonomous differential equation

$$\frac{dx}{dt} = (x^2 - 5x + 4)(x^2 - 16)$$

and determine which critical points are stable and unstable.

$$\left.\begin{aligned} 0 &= (x^2-5x+4) = (x-4)(x-1) \\ 0 &= (x^2-16) = (x-4)(x+4) \end{aligned}\right\} \Rightarrow \text{Crit. pts @ } x = 1, 4, 4, -4.$$

Phase Diagram



2. (10 points) Consider a rabbit population satisfying the logistic equation

$$\frac{dP}{dt} = 2P - (0.005)P^2,$$

where t is measured in years. If the initial population is 700 rabbits, how many months does it take for $P(t)$ to reach 105% of its limiting population M?

$$\frac{dP}{dt} = 2P-(0.005)P^2 = 0.005P(400-P)$$

Logistic Eqn: $P(t) = \dfrac{400\cdot 700}{700+(-300)e^{-2t}}$

105% of $400 = 420$

$$420 = \frac{280{,}000}{700+(-300)e^{-2t}}$$

$$(-300)e^{-2t} = \frac{280{,}000}{420} - 700 = -33.\overline{3}$$

$$e^{-2t} = 0.\overline{1}$$

$$-2t = \ln(0.\overline{1}) \approx -2.197$$

$t \approx 1.099$ yrs or 13.18 months

3. (3 points) Recall that an object's velocity (moving vertically) is given by

$$\frac{dv}{dt} = -g - \rho v^p,$$

where g is the force of gravity, $\rho = \frac{k}{m} > 0$, and $1 \leq p \leq 2$. Suppose a team of scientists are trying to determine a projectile's escape velocity from Earth's atmosphere. That team of scientists makes the assumption that $p = 2$ and finds that the initial velocity required to escape Earth's atmosphere (without additional thrust) is given by

$$v_0 = \sqrt{\frac{2GM}{R}}$$

where M is the mass of the Earth and R is its equatorial radius. Give a sentence of justification as to why this initial velocity will be sufficient to escape Earth's atmosphere for all values of p.

This is sufficient because $p=2$ assumes the largest amount of air resistance, meaning for $p<2$, the initial velocity would be less than $\sqrt{\frac{2GM}{R}}$.

4. (7 points) Consider a body that moves horizontally through a medium whose resistance is given by

$$\frac{dv}{dt} = -2v^{3/2}.$$

Assuming that $v(0) = 1$ and $x(0) = 1$, find the position $x(t)$ as a function of t.

$$\frac{dv}{dt} = -2v^{3/2}$$

$$\int \frac{dv}{v^{3/2}} = \int -2\,dt$$

$$-2v^{-1/2} = -2t + C$$

w/ $v(0)=1$, $-2 = C$

So $v^{-1/2} = t+1 \Rightarrow v = \frac{1}{(t+1)^2}$.

Then $x(t) = \int \frac{1}{(t+1)^2}\,dt = \frac{-1}{t+1} + C$.

With $x(0) = 1 = -1 + C \Rightarrow C = 2$.

So $x(t) = \frac{-1}{t+1} + 2$.

5. (10 points) Find the general solution of the differential equation

$$6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4 = 0$$

which has characteristic function

$$(r^2 + 4)(6r^2 + 5r + 1) = 0.$$

$r = \pm 2i$

$6r^2 + 3r + 2r + 1 = 0$

$3r(2r+1) + 2r + 1 = 0$

$(3r+1)(2r+1) = 0$

$r = -\frac{1}{2}, -\frac{1}{3}.$

So $y = c_1 e^{-1/2 t} + c_2 e^{-1/3 t} + (a_1 \cos 2t + b_1 \sin 2t)$

6. (10 points) A 8-lb weight (mass m=0.25 slugs) is attached both to a vertically suspended spring that it stretches ~~3 in.~~ 12in. and to a dashpot that provides 2 lb of resistance for every foot per second of velocity.

(a) The weight is pushed up 6 in above its static equilibrium position and then released from rest at time $t = 0$, find its position function $x(t)$.

(b) Determine if the motion is over-damped, critically damped or under-damped.

Hint: If you can not figure out the constants, make a guess and do the rest of the problem to demonstrate your ability to do other aspects of the problem for partial credit.

$m = 0.25$, $K = \frac{8}{1} = 8$, $c = 2$

So $mx'' + cx' + Kx = 0 \Rightarrow 0.25x'' + 2x' + 8x = 0 \Rightarrow x'' + 8x' + 32x = 0$

<u>Char Eqn:</u> $r^2 + 8r + 32 = 0 \Rightarrow r_{1,2} = \frac{-8 \pm \sqrt{8^2 - 4 \cdot 32}}{2} = -4 \pm \frac{\sqrt{-64}}{2} = -4 \pm i4$

So $x = e^{-4t}(A\cos 4t + B\sin 4t)$ and we have <u>under-damped motion.</u>

We also have $x_0 = -\frac{1}{2}$ and $v_0 = 0$.

So $x_0 = \frac{1}{2} = A$. $v_0 = 0 = \Big[-4e^{-4t}\left(\frac{1}{2}\cos 4t + B\sin 4t\right) + e^{-4t}(-2\sin 4t + 4B\cos 4t)\Big]\Big|_{t=0}$

$= -2 + 4B \Rightarrow B = \frac{1}{2}$.

Therefore $x(t) = \frac{e^{-4t}}{2}(\cos 4t + \sin 4t)$.